Comment on "Orthogonality of Generally Normalized Eigenvectors and Eigenrows"

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To avoid repetition, equations used below from Ref. 1 have their designation prefixed by an R. The development in Ref. 1 is clear down to the end of Eq. (R8). However when the generalized normality condition, Eq. (R9), is used in place of Eq. (R5), the consequent generalization of Eq. (R8) to become Eq. (R12) can be proved in a much more straightforward manner than presented.[1]

A straightforward proof is as follows: Using the normalization conditions of Eq. (R9), and the M defined by Eq. (R11), one need only define two new matrices, as follows:

$$P=[p_1 p_2 \ldots p_{2n}] \tag{1}$$

$$Q^T=[q_1 q_2 \ldots q_{2n}] \tag{2}$$

where

$$\left.\begin{array}{l} p_r=x_r/m_r^{1/2} \\ q_r=y_r/m_r^{1/2} \end{array}\right\} \quad (r=1,2,\ldots,2n) \tag{3}$$

There is no difficulty with the $m_r^{-1/2}$ factors since none of the m_r can be zero in Eq. (R9). In terms of the X and Y matrices,[1] Eq. (3) becomes

$$P=X\,M^{-1/2}$$
$$Q=M^{-1/2}\,Y \tag{4}$$

Noting that the orthogonality conditions for x_r and y_r are identical for p_r and q_r, and that the normality conditions, Eq. (R9), become

$$q_r^T[2\lambda_r A+B]p_r=1 \tag{5}$$

the matrices P and Q must satisfy Eqs. (R6) and (R8), viz.,

$$\Lambda QAP+QAP\Lambda+QBP=I \tag{6}$$

and

$$PQ=0 \tag{7}$$

From Eq. (4), it can be seen that these equations, written in terms of X and Y are just Eqs. (R10) and (R12).

The following related result may also be of interest although it is not mentioned in Ref. 1.

$$YCX=-M\Lambda+\Lambda YAX\Lambda=-\Lambda M+\Lambda YAX\Lambda \tag{8}$$

Reference

[1] Fawzy, I., "Orthogonality of Generally Normalized Eigenvectors and Eigenrows," *AIAA Journal,* Vol. 15, Feb. 1977, pp. 276-278.

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Comment on "Generalized Inverse of a Matrix: The Minimization Approach"

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Shapiro and Decarli[1] have given a useful summary of the application of the generalized inverse of a matrix to the solution of a set of linear equations. Although they have given the solution for the extended vector norm, there are some additional points concerning the case of overdetermined sets of equations that may be mentioned usefully.

We consider the algebraic matrix equation:

$$y=Ax \tag{1}$$

where y and x are vectors of dimensions n and m, respectively, and A is a constant matrix. We wish to determine x when y and A are given. In particular, we consider the case where $n>m$ and A has full rank, the equations are inconsistent, and we seek a least-squares solution.

Premultiplying Eq. (1) on both sides by a diagonal matrix of positive numbers R, we have the same set of equations, but now weighted so as to represent the relative importance attached to the individual equations. Thus, we have:

$$Ry=RAx \tag{2}$$

the solution of which, using the generalized inverse, is:

$$x=(A^TR^2A)^{-1}A^TR^2y \tag{3}$$

Equation (3) clearly represents a special case of the authors' Eq. (21), where $Q=R^2$; but here we have a direct interpretation of the significance of R. The interpretation of a more general, nondiagonal matrix Q is more obscure.

From a practical point of view, the way in which Eq. (1) are formulated in an engineering problem is usually haphazard, with respect to constant multipliers of the rows. We may have multiplied throughout by any number at some stage without considering the effect of this on the weighting of the equations. It follows that we should always give consideration to the inclusion of the weighting matrix R. Correspondingly, references in the literature to *the* least-squares solution should often more properly be to *a* least-squares solution. Values to be given to the elements of R may not be known at the outset, but can be obtained by trial, on consideration of how well each equation of the set is satisfied by a least-squares solution.

Reference

[1] Shapiro, E.Y., and Decarli, H.E., "Generalized Inverse of a Matrix: The Minimization Approach," *AIAA Journal,* Vol. 14, Oct. 1976, pp. 1483-1484.

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